Qualifying Exam, Quantum Mechanics, 2020 July

1. (70 pts) Estimations related to the Hydrogen atom in the ground state(Use values of the electron mass, $m_e \simeq 10^{-30}\ kg$, Planck constant $\hbar \simeq 10^{-34} J \cdot s$, and the charge of electron, $e \simeq -10^{-19} C \simeq -1 \times 10^{-14}\sqrt{J \cdot m}$.)

   (a) Estimate the size of the Hydrogen atom (in the ground state)

   (b) Estimate the velocity of the electron in the Hydrogen atom

   (c) Estimate the magnetic dipole moment of the electron in the Hydrogen atom (consider the orbital motion)

   (d) Estimate the electric field inside the Hydrogen atom

   (e) Estimate the magnetic field inside the Hydrogen atom(due to orbital motion of the electron)

   (f) Estimate the energy due to spin-orbit interaction (you may assume the magnetic dipole moment due to spin is similar to the one due to orbital motion)

   (g) Estimate the ratio of the energy due spin-orbit interaction and the ground state energy

2. (60 pts) Consider the angular momentum operators in 3-D,

$$[\hat{J}_l, \hat{J}_k] = i\hbar \sum_{n=1}^{3} \epsilon_{lkn} \hat{J}_n \tag{1}$$

   where $l, k, n = 1, 2, 3$ and ϵ_{lkn} is a totally antisymmetry tensor with $\epsilon_{123} = 1$

   (a) Consider the orthonormal eigenstates, $|j, m\rangle$'s which are defined to satisfy $\hat{J}^2|j, m\rangle = j(j+1)\hbar^2|j, m\rangle$ and $\hat{J}_3|j, m\rangle = m\hbar|j, m\rangle$ with $j = 0, 1/2, 1, 3/2..$ and $|m| \leq j$ are also half integers. If we define $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$. Show that

$$\hat{J}_\pm|j, m\rangle = \hbar\sqrt{j(j+1) - m^2 \mp m}|j, m \pm 1\rangle \tag{2}$$

   (b) Consider the spin 1/2 particle which is described by the $j = 1/2$ states. Let the Hamiltonian of the particle to be $\hat{H} = \hat{a}J_1$, where a is real. Find the energy eigenstates and eigenvalues.

   (c) Continue the problem in 2-(b), if we prepare the particle initially in the state $|\psi(0)\rangle = |\frac{1}{2} - \frac{1}{2}\rangle$, find the state at the time t.

   (d) Continue the problem in 2-(c), find the expectation value of the spin in the 2-direction at the time t.

   (e) Find the states $|+\rangle = \hat{R}_1(\frac{\pi}{2})|\frac{1}{2}\frac{1}{2}\rangle$ and $|-\rangle = \hat{R}_1(-\frac{\pi}{2})|\frac{1}{2}\frac{1}{2}\rangle$. Where $\hat{R}_1(\theta) = e^{\frac{i}{\hbar}\hat{J}_1\theta}$ is the rotational operator along 1-direction by angle θ.

   (f) Show that $|+\rangle$ and $|-\rangle$ are eigenstates of $\hat{J}_2$

3. (30 pts) Consider two spin-$\frac{1}{2}$ particles and define the product states $|\uparrow\uparrow\rangle \equiv |\frac{1}{2}\frac{1}{2}\rangle|\frac{1}{2}\frac{1}{2}\rangle, |\uparrow\downarrow\rangle \equiv |\frac{1}{2}\frac{1}{2}\rangle|\frac{1}{2}-\frac{1}{2}\rangle, |\downarrow\uparrow\rangle \equiv |\frac{1}{2}-\frac{1}{2}\rangle|\frac{1}{2}\frac{1}{2}\rangle, |\downarrow\downarrow\rangle \equiv |\frac{1}{2}-\frac{1}{2}\rangle|\frac{1}{2}-\frac{1}{2}\rangle$

   (a) Show that the state $|\phi_1\rangle = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ is a singlet state (total angular momentum is zero)

   (b) Show that the singlet state can also be written as $|\phi_1\rangle = |+\rangle|-\rangle - |-\rangle|+\rangle$($|+\rangle$ and $|-\rangle$ are defined in problem 2-(e))

   (c) Show that the state $|\phi_2\rangle = |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$ is a component of a triplet state (total angular momentum is one)

4. (90 pts) Consider a 1D simple harmonic oscillator with the Hamiltonian, $\hat{H}_{sho} = \frac{\hat{p}^2}{2M} + \frac{1}{2}M\omega^2\hat{x}^2$. The Hilbert space is defined by a set of basis vectors $|x\rangle$'s with $\hat{x}|x\rangle = x|x\rangle$ (x is real) and the inner product defined as $\langle x'|x\rangle = \delta(x'-x)$. And $\hat{p}|x\rangle = -i\hbar\frac{\partial}{\partial x}|x\rangle$.

   (a) Show that $[\hat{x}, \hat{p}] = i\hbar\hat{1}$.

   (b) Show that $\hat{p}$ is a hermitian operator.

   (c) If we define $\hat{a} = \frac{1}{\sqrt{2M\hbar\omega}}(\hat{p} - iM\omega x)$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2M\hbar\omega}}(\hat{p} + iM\omega x)$, show that $[\hat{a}, \hat{a}^\dagger] = \hat{1}$, $[\hat{H}_{sho}, \hat{a}] = -\hbar\omega\hat{a}$ and $[\hat{H}_{sho}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger$

   (d) Find the normalized wavefunction, $|\psi_0\rangle$ that satisfies $\hat{a}|\psi_0\rangle = 0$.

   (e) Show that $|\psi_n\rangle \equiv (\hat{a}^\dagger)^n|\psi_0\rangle$ is a energy eigenstate of $\hat{H}_{sho}$ with eigenvalue $\hbar\omega(n+\frac{1}{2})$.

   (f) Show that $\langle\psi_m|\psi_n\rangle = n!\delta_{mn}$

   (g) Add a small perturbation $\delta\hat{H} = C\hat{x}^4$ to $\hat{H}_{sho}$. Use the first order perturbation theory to calculate the energy shift for each energy eigenstate $|\psi_2\rangle$.

   (h) Use the WKB approximation,

$$\int_{a_E}^{b_E} k(x')dx' = (n+\frac{1}{2})\pi \tag{3}$$

   (k is the classical wavevector) to calculate the bound states energies for $\hat{H}_{sho}$.

   (i) Use the variation method with the trial wavefunction $\langle x|\psi(\lambda)\rangle = e^{-\lambda x^2}$ to find the ground state energy and wavefunction for $\hat{H}_{sho}$ by variating λ